

Generalizations of Quantum Statistics

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Abstract

We review generalizations of quantum statistics, including parabose, parafermi, and quon statistics, but not including anyon statistics, which is special to two dimensions.

The general principles of quantum theory allow statistics more general than bosons or fermions. (Bose statistics and Fermi statistics are discussed in separate articles.) The restriction to Bosons or Fermions requires the symmetrization postulate, “the states of a system containing N identical particles are necessarily either all symmetric or all antisymmetric under permutations of the N particles,” or, equivalently, “all states of identical particles are in one-dimensional representations of the symmetric group [1].” A.M.L. Messiah and O.W. Greenberg discussed quantum mechanics without the symmetrization postulate [2]. The spin-statistics connection, that integer spin particles are bosons and odd-half-integer spin particles are fermions [3], is an independent statement. Identical particles in 2 space dimensions are a special case, “anyons.” (Anyons are discussed in a separate article.) Braid group statistics, a nonabelian analog of anyons, are also special to 2 space dimensions

All observables must be symmetric in the dynamical variables associated with identical particles. Observables can not change the permutation symmetry type of the wave function; i.e. there is a superselection rule separating states in inequivalent representations of the symmetric group and when identical particles can occur in states that violate the spin-statistics connection their transitions must occur in the same representation of the symmetric group. One can not introduce a small violation

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of statistics by assuming the Hamiltonian is the sum of a statistics-conserving and a small statistics-violating term, $H = H_S + \epsilon H_V$, as one can for violations of parity, charge conjugation, etc. Violation of statistics must be introduced in a more subtle way.

S. Doplicher, R. Haag and J. Roberts [4] classified identical particle statistics in 3 or more space dimensions. They found parabose and parafermi statistics of positive integer orders, which had been introduced by H.S. Green [5], and infinite statistics, which had been introduced by Greenberg [6]. Parabose (parafermi) statistics allows up to p identical particles in an antisymmetric state (symmetric) state. Infinite statistics allows any number of identical particles in a symmetric or antisymmetric state.

Trilinear commutation relations,

$$[[a_k^\dagger, a_l]_\pm, a_m^\dagger]_- = 2\delta_{lm}a_k^\dagger \quad (1)$$

with the vacuum condition, $a_k|0\rangle = 0$, and single-particle condition, $a_k a_l^\dagger|0\rangle = p\delta_{kl}|0\rangle$, define the Fock representation of order p parabose (parafermi) statistics. Green found two infinite sets of solutions of these commutation rules, one set for each positive integer p , by the ansatz,

$$a_k^\dagger = \sum_{\alpha=1}^p b_k^{(\alpha)\dagger}, \quad a_k = \sum_{\alpha=1}^p b_k^{(\alpha)}, \quad (2)$$

where the $b_k^{(\alpha)}$ and $b_k^{(\beta)\dagger}$ are bose (fermi) operators for $\alpha = \beta$ but anticommute (commute) for $\alpha \neq \beta$ for the parabose (parafermi) cases. The integer p is the order of the parastatistics. For parabosons (parafermions) p is the maximum number of particles that can occupy an antisymmetric (symmetric) state. The case $p = 1$ corresponds to the usual Bose or Fermi statistics. Greenberg and Messiah [7] proved that Green's ansatz gives all Fock-like solutions of Green's commutation rules. Local observables in parastatistics have a form analogous to the usual ones; for example, the local current for a spin-1/2 theory is $j_\mu = (1/2)[\bar{\psi}(x), \psi(x)]_-$. From Green's ansatz, it is clear that the squares of all norms of states are positive; thus parastatistics [8] gives a set of orthodox positive metric theories. Parabose or parafermi statistics for $p > 1$ give gross violations of Bose or Fermi statistics so that parastatistics theories are not useful to parametrize small violations of statistics.

The bilinear commutation relation

$$a(k)a^\dagger(l) - qa^\dagger(l)a(k) = \delta(k, l), \quad (3)$$

with the vacuum condition, $a(k)|0\rangle = 0$, define the Fock representation of quon statistics. Positivity of norms requires $-1 \leq q \leq 1$ [9, 10]. Outside this range the squared norms become negative. There is no commutation relation involving two a 's or two a^\dagger 's. There are $n!$ linearly independent n -particle states in Hilbert space if all quantum numbers are distinct; these states differ only by permutations of the order of the creation operators.

For $q \approx \pm 1$, quons provide a formalism that can parametrize small violations of statistics so that quons are useful for quantitative tests of statistics. At $q = 1(-1)$ only the symmetric (antisymmetric) representation of \mathcal{S}_n occurs. The quon operators interpolate smoothly between fermi and bose statistics in the sense that as $q \rightarrow \mp 1$ the antisymmetric (symmetric) representations smoothly become more heavily weighted.

Although there are $n!$ linearly independent vectors in Fock space associated with a degree n monomial in creation operators that carry disjoint quantum numbers acting on the vacuum, there are fewer than $n!$ observables associated with such vectors. The general observable is a linear combination of projectors on the irreducibles of the symmetric group.

A convenient way to parametrize violations or bounds on violations of statistics uses the two-particle density matrix. For fermions, $\rho_2 = (1 - v_F)\rho_a + v_F\rho_s$; for bosons, $\rho_2 = (1 - v_B)\rho_s + v_B\rho_a$. In each case the violation parameter varies between zero if the statistics is not violated and one if the statistics is completely violated. R.C. Hilborn [11] pointed out that the transition matrix elements between symmetric (antisymmetric) states are proportional to $(1 \pm q)$ so that the transition probabilities are proportional to $(1 \pm q)^2$ rather than to $(1 \pm q)$.

Several properties of kinematically relativistic quon theories hold, including a generalization of Wick's theorem, cluster decomposition theorems and (at least for free quon fields) the *CPT* theorem; however locality in the sense of the commutativity of observables at spacelike separation fails [6]. The nonrelativistic form of locality

$$[\rho(x), \psi^\dagger(y)]_- = \delta(x - y)\psi^\dagger(y), \quad (4)$$

where ρ is the charge density, does hold.

Greenberg and Hilborn [12] derived the generalization of the result due to E.P. Wigner [13] and to P. Ehrenfest and J.R. Oppenheimer [14] that a bound state of bosons and fermions is a boson unless it has an odd number of fermions, in which case it is a fermion generalizes for quons: A bound state of n identical quons with parameter $q_{\text{constituent}}$ has parameter $q_{\text{bound}} = q_{\text{constituent}}^{n^2}$ [12].

Note: References [1] through [14] are primary references. References [15] through [18] are secondary references.

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